

KOBAYASHI–ROYDEN VS. HAHN PSEUDOMETRIC IN \mathbb{C}^2

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ABSTRACT. For a domain $D \subset \mathbb{C}$ the Kobayashi–Royden \varkappa and Hahn h pseudometrics are equal iff D is simply connected. Overholt showed that for $D \subset \mathbb{C}^n$, $n \geq 3$, we have $h_D \equiv \varkappa_D$. Let $D_1, D_2 \subset \mathbb{C}$. The aim of this paper is to show that $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$ iff at least one of D_1, D_2 is simply connected or biholomorphic to $\mathbb{C} \setminus \{0\}$. In particular, there are domains $D \subset \mathbb{C}^2$ for which $h_D \not\equiv \varkappa_D$.

1. Introduction.

For a domain $D \subset \mathbb{C}^n$, the Kobayashi–Royden pseudometric \varkappa_D and the Hahn pseudometric h_D are defined by the formulas:

$$\begin{aligned}\varkappa_D(z; X) &:= \inf\{|\alpha| : \exists f \in \mathcal{O}(E, D) \ f(0) = z, \ \alpha f'(0) = X\}, \\ h_D(z; X) &:= \inf\{|\alpha| : \exists f \in \mathcal{O}(E, D) \ f(0) = z, \ \alpha f'(0) = X, \ f \text{ is injective}\}, \\ &\quad z \in D, X \in \mathbb{C}^n,\end{aligned}$$

where E denotes the unit disc (cf. [Roy], [Hah], [Jar-Pfl]). Obviously $\varkappa_D \leq h_D$. It is known that both pseudometrics are invariant under biholomorphic mappings, i.e., if $f: D \rightarrow \tilde{D}$ is biholomorphic, then

$$h_D(z; X) = h_{\tilde{D}}(f(z); f'(z)(X)), \quad \varkappa_D(z; X) = \varkappa_{\tilde{D}}(f(z); f'(z)(X)), \\ z \in D, X \in \mathbb{C}^n.$$

It is also known that for a domain $D \subset \mathbb{C}$ we have: $h_D \equiv \varkappa_D$ iff D is simply connected. In particular $h_D \not\equiv \varkappa_D$ for $D = \mathbb{C}_* := \mathbb{C} \setminus \{0\}$. It has turned out that $h_D \equiv \varkappa_D$ for any domain $D \subset \mathbb{C}^n$, $n \geq 3$ ([Ove]). The case $n = 2$ was investigated for instance in [Hah], [Ves], [Vig], [Cho], but neither a proof nor a counterexample for the equality was found (existing ‘counterexamples’ were based on incorrect product properties of the Hahn pseudometric).

2. The main result.

Theorem 1. *Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then:*

1. *If at least one of D_1, D_2 is simply connected, then $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$.*
2. *If at least one of D_1, D_2 is biholomorphic to \mathbb{C}_* , then $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$.*

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3. Otherwise $h_{D_1 \times D_2} \not\equiv \kappa_{D_1 \times D_2}$.

Let $p_j: D_j^* \rightarrow D_j$ be a holomorphic universal covering of D_j ($D_j^* \in \{\mathbb{C}, E\}$), $j = 1, 2$. Recall that if D_j is simply connected, then $h_{D_j} \equiv \kappa_{D_j}$. If D_j is not simply connected and D_j is not biholomorphic to \mathbb{C}_* , then, by the uniformization theorem, $D_j^* = E$ and p_j is not injective.

Hence, Theorem 1 is an immediate consequence of the following three propositions (we keep the above notation).

Proposition 2. *If $h_{D_1} \equiv \kappa_{D_1}$, then $h_{D_1 \times D_2} \equiv \kappa_{D_1 \times D_2}$ for any domain $D_2 \subset \mathbb{C}$.*

Proposition 3. *If D_1 is biholomorphic to \mathbb{C}_* , then $h_{D_1 \times D_2} \equiv \kappa_{D_1 \times D_2}$ for any domain $D_2 \subset \mathbb{C}$.*

Proposition 4. *If $D_j^* = E$ and p_j is not injective, $j = 1, 2$, then $h_{D_1 \times D_2} \not\equiv \kappa_{D_1 \times D_2}$.*

Observe the following property that will be helpful in proving the propositions.

Remark 5. *For any domain $D \subset \mathbb{C}^n$ we have $h_D \equiv \kappa_D$ iff for any $f \in \mathcal{O}(E, D)$, $\vartheta \in (0, 1)$ with $f'(0) \neq 0$, there exists an injective $g \in \mathcal{O}(E, D)$ such that $g(0) = f(0)$ and $g'(0) = \vartheta f'(0)$.*

Proof of Proposition 2. Let $f = (f_1, f_2) \in \mathcal{O}(E, D_1 \times D_2)$ and let $\vartheta \in (0, 1)$.

First, consider the case where $f'_1(0) \neq 0$.

By Remark 5, there exists an injective function $g_1 \in \mathcal{O}(E, D_1)$ such that $g_1(0) = f_1(0)$ and $g'_1(0) = \vartheta f'_1(0)$. Put $g(z) := (g_1(z), f_2(\vartheta z))$.

Obviously $g \in \mathcal{O}(E, D_1 \times D_2)$ and g is injective. Moreover, $g(0) = f(0)$ and $g'(0) = (g'_1(0), f'_2(0)\vartheta) = (\vartheta f'_1(0), \vartheta f'_2(0)) = \vartheta f'(0)$.

Suppose now that $f'_1(0) = 0$. Take $0 < d < \text{dist}(f_1(0), \partial D_1)$ and put

$$h(z) := \frac{f_2(\vartheta z) - f_2(0)}{f'_2(0)}, \quad M := \max\{|h(z)| : z \in \overline{E}\},$$

$$g_1(z) := f_1(0) + \frac{d}{M+1}(h(z) - \vartheta z), \quad g(z) := (g_1(z), f_2(\vartheta z)), \quad z \in E.$$

Obviously $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $|g_1(z) - f_1(0)| < d$, we get $g_1(z) \in B(f_1(0), d) \subset D_1$, $z \in E$. Hence $g \in \mathcal{O}(E, D_1 \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally $g(0) = (g_1(0), f_2(0)) = (f_1(0) + \frac{d}{M+1}h(0), f_2(0)) = f(0)$ and $g'(0) = (g'_1(0), \vartheta f'_2(0)) = (\frac{d}{M+1}(h'(0) - \vartheta), \vartheta f'_2(0)) = \vartheta f'(0)$. \square

Proof of Proposition 3. We may assume that $D_1 = \mathbb{C}_*$ and $D_2 \neq \mathbb{C}$. Using Remark 5, let $f = (f_1, f_2) \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$ and let $\vartheta \in (0, 1)$. Applying an appropriate automorphism of \mathbb{C}_* , we may assume that $f_1(0) = 1$.

For the case where $f'_2(0) = 0$, we apply the above construction to the domains $\tilde{D}_1 = f_2(0) + \text{dist}(f_2(0), \partial D_2)E$, $\tilde{D}_2 = \mathbb{C}_*$ and mappings $\tilde{f}_1 \equiv f_2(0)$, $\tilde{f}_2 = f_1$.

Now, consider the case where $f'_2(0) \neq 0$ and $\vartheta f'_1(0) = 1$. We put

$$g_1(z) := 1 + z, \quad g(z) := (g_1(z), f_2(\vartheta z)), \quad z \in E.$$

Obviously, $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$ and g is injective. We have $g(0) = (1, f_2(0)) = f(0)$ and $g'(0) = (1, \vartheta f'_2(0)) = \vartheta f'(0)$.

In all other cases, let $M := \max\{|f_2(z)| : |z| \leq \vartheta\}$. Take a $k \in \mathbb{N}$ such that $|c_k| > M$, where

$$c_k := f_2(0) - k \frac{\vartheta f'_2(0)}{\vartheta f'_1(0) - 1}.$$

Put

$$h(z) := \frac{f_2(\vartheta z) - c_k}{f_2(0) - c_k},$$

$$g_1(z) := (1 + z)h^k(z), \quad g_2(z) := f_2(\vartheta z), \quad g(z) := (g_1(z), g_2(z)), \quad z \in E.$$

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $h(z) \neq 0$, we have $g_1(z) \neq 0$, $z \in E$. Hence $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally $g(0) = (h^k(0), f_2(0)) = f(0)$ and

$$\begin{aligned} g'(0) &= (g'_1(0), \vartheta f'_2(0)) = (h^k(0) + kh^{k-1}(0)h'(0), \vartheta f'_2(0)) \\ &= \left(1 + k \frac{\vartheta f'_2(0)}{f_2(0) - c_k}, \vartheta f'_2(0)\right) = (1 + \vartheta f'_1(0) - 1, \vartheta f'_2(0)) = \vartheta f'(0). \end{aligned}$$

□

Proof of Proposition 4. It suffices to show that there exist $\varphi_1, \varphi_2 \in \text{Aut}(E)$ and a point $q = (q_1, q_2) \in E^2$, $q_1 \neq q_2$, such that $p_j(\varphi_j(q_1)) = p_j(\varphi_j(q_2))$, $j = 1, 2$, and $\det[(p_j \circ \varphi_j)'(q_k)]_{j,k=1,2} \neq 0$.

Indeed, put $\tilde{p}_j := p_j \circ \varphi_j$, $j = 1, 2$, and suppose that $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$. Put $a := (\tilde{p}_1(0), \tilde{p}_2(0))$ and $X := (\tilde{p}'_1(0), \tilde{p}'_2(0)) \in (\mathbb{C}_*)^2$. Take an arbitrary $f \in \mathcal{O}(E, D_j)$ with $f(0) = a_j$. Let \tilde{f} be the lifting of f with respect to \tilde{p}_j such that $\tilde{f}(0) = 0$. Since $|\tilde{f}'(0)| \leq 1$, we get $|f'(0)| \leq |X_j|$. Consequently $\varkappa_{D_j}(a_j; X_j) = 1$, $j = 1, 2$. In particular, $\varkappa_{D_1 \times D_2}(a; X) = \max\{\varkappa_{D_1}(a_1; X_1), \varkappa_{D_2}(a_2; X_2)\} = 1$.

Let $(0, 1) \ni \alpha_n \nearrow 1$. Fix an $n \in \mathbb{N}$. Since $\varkappa_{D_1 \times D_2}(a; X) = 1$, there exists $f_n \in \mathcal{O}(E, D_1 \times D_2)$ such that $f_n(0) = a$ and $f'_n(0) = \alpha_n X$. By Remark 5, there exists an injective holomorphic mapping $g_n = (g_{n,1}, g_{n,2}) : E \rightarrow D_1 \times D_2$ such that $g_n(0) = a$ and $g'_n(0) = \alpha_n^2 X$. Let $\tilde{g}_{n,j}$ be the lifting with respect to \tilde{p}_j of $g_{n,j}$ with $\tilde{g}_{n,j}(0) = 0$, $j = 1, 2$.

By the Montel theorem, we may assume that the sequence $(\tilde{g}_{n,j})_{n=1}^\infty$ is locally uniformly convergent, $\tilde{g}_{0,j} := \lim_{n \rightarrow \infty} \tilde{g}_{n,j}$. We have $\tilde{g}'_{0,j}(0) = 1$, $\tilde{g}_{0,j} : E \rightarrow E$. By the Schwarz lemma we have $\tilde{g}_{0,j} = \text{id}_E$, $j = 1, 2$.

Let $h_{0,j}(z_1, z_2) := \tilde{p}_j(z_1) - \tilde{p}_j(z_2)$, $(z_1, z_2) \in E^2$,

$$V_j = V(h_{0,j}) = \{(z_1, z_2) \in E^2 : h_{0,j}(z_1, z_2) = 0\}, \quad j = 1, 2.$$

Since

$$\det \left[\frac{\partial h_{0,j}}{\partial z_k}(q) \right]_{j,k=1,2} = -\det [\tilde{p}'_j(q_k)]_{j,k=1,2} \neq 0,$$

V_1 and V_2 intersect transversally at q . Let $U \subset \subset \{(z_1, z_2) \in E^2 : z_1 \neq z_2\}$ be a neighborhood of q such that $V_1 \cap V_2 \cap \overline{U} = \{q\}$. For $n \in \mathbb{N}$, $j = 1, 2$, define

$$h_{n,j}(z_1, z_2) := g_{n,j}(z_1) - g_{n,j}(z_2), \quad (z_1, z_2) \in E^2.$$

Observe that the sequence $(h_{n,j})_{n=1}^\infty$ converges uniformly on \overline{U} to $h_{0,j}$, $j = 1, 2$. In particular (cf. [Two-Win]), we have $V(h_{n,1}) \cap V(h_{n,2}) \cap \overline{U} = \{z \in \overline{U} : h_{n,1}(z) = h_{n,2}(z) = 0\} \neq \emptyset$ for some $n \in \mathbb{N}$ — contradiction.

We move now to the construction of φ_1, φ_2 and q . Let $\psi_j \in \text{Aut}(E)$ be a non-identity lifting of p_j with respect to p_j ($p_j \circ \psi_j \equiv p_j$, $\psi_j \not\equiv \text{id}$), $j = 1, 2$. Observe that ψ_j has no fixed points (a lifting is uniquely determined by its value at one point), $j = 1, 2$.

To simplify notation, let

$$h_a(z) := \frac{z - a}{1 - \overline{a}z}, \quad a, z \in E.$$

One can easily check that

$$\sup_{z \in E} m(z, \psi_j(z)) = 1, \quad j = 1, 2,$$

where $m(z, w) := |h_w(z)| = \left| \frac{z-w}{1-\overline{z}w} \right|$ is the Möbius distance. Hence there exist $\varepsilon \in (0, 1)$ and $z_1, z_2 \in E$ with $m(z_1, \psi_1(z_1)) = m(z_2, \psi_2(z_2)) = 1 - \varepsilon$. Let $d \in (0, 1)$, $h_1, h_2 \in \text{Aut}(E)$ be such that $h_j(-d) = z_j$, $h_j(d) = \psi_j(z_j)$, $j = 1, 2$.

If $(p_j \circ h_j)'(-d) \neq \pm(p_j \circ h_j)'(d)$ for some j (we may assume that for $j = 1$), then at least one of the determinants

$$\det \begin{bmatrix} (p_1 \circ h_1)'(-d), & (p_1 \circ h_1)'(d) \\ (p_2 \circ h_2)'(-d), & (p_2 \circ h_2)'(d) \end{bmatrix},$$

$$\det \begin{bmatrix} (p_1 \circ h_1 \circ (-\text{id}))'(-d), & (p_1 \circ h_1 \circ (-\text{id}))'(d) \\ (p_2 \circ h_2)'(-d), & (p_2 \circ h_2)'(d) \end{bmatrix},$$

is nonzero.

Otherwise, let $\tilde{\psi}_j = h_j^{-1} \circ \psi_j \circ h_j$ and $\tilde{p}_j = p_j \circ h_j$, $j = 1, 2$. Observe that $\tilde{\psi}_j(-d) = d$ and $(\tilde{\psi}'_j(-d))^2 = 1$, $j = 1, 2$. Thus, each $\tilde{\psi}_j$ is either $-\text{id}$ or h_c , where $c = -\frac{2d}{1+d^2}$. The case $\tilde{\psi}_j = -\text{id}$ is impossible since $\tilde{\psi}_j$ has no fixed points. By

substituting p_j by \tilde{p}_j and ψ_j by $\tilde{\psi}_j$, $j = 1, 2$, the proof reduces to the case, where $\psi_1 = \psi_2 = h_c =: \psi$ for some $-1 < c < 0$.

We claim that there exists a point $a \in E$ such that if an automorphism $\varphi = \varphi_a \in \text{Aut}(E)$ satisfies $\varphi(a) = \psi(a)$ and $\varphi(\psi(a)) = a$, then $\varphi'(a) \neq \pm\psi'(a)$. Suppose for a moment that such an a has been found. Notice that $\varphi \circ \varphi = \text{id}$ and hence $\varphi'(\psi(a)) = \frac{1}{\varphi'(a)}$. Put $\varphi_1 := \text{id}$, $\varphi_2 := \varphi$, $q := (a, \psi(a))$. We have

$$\begin{aligned} & \det \begin{bmatrix} (p_1 \circ \varphi_1)'(a), & (p_1 \circ \varphi_1)'(\psi(a)) \\ (p_2 \circ \varphi_2)'(a), & (p_2 \circ \varphi_2)'(\psi(a)) \end{bmatrix} \\ = & \det \begin{bmatrix} p_1'(a), & p_1'(\psi(a)) \\ p_2'(\varphi(a))\varphi'(a), & p_2'(\varphi(\psi(a)))\varphi'(\psi(a)) \end{bmatrix} \\ = & \det \begin{bmatrix} (p_1 \circ \psi)'(a), & p_1'(\psi(a)) \\ p_2'(\psi(a))\varphi'(a), & (p_2 \circ \psi)'(a)\frac{1}{\varphi'(a)} \end{bmatrix} \\ = & \det \begin{bmatrix} p_1'(\psi(a))\psi'(a), & p_1'(\psi(a)) \\ p_2'(\psi(a))\varphi'(a), & p_2'(\psi(a))\psi'(a)\frac{1}{\varphi'(a)} \end{bmatrix} \\ = & p_1'(\psi(a))p_2'(\psi(a)) \det \begin{bmatrix} \psi'(a), & 1 \\ \varphi'(a), & \frac{\psi'(a)}{\varphi'(a)} \end{bmatrix} \neq 0, \end{aligned}$$

which finishes the construction.

It remains to find a . First observe that the equality $\varphi'_a(a) = \psi'(a)$ is impossible. Otherwise $\varphi_a = \psi$ and consequently $\psi \circ \psi = \text{id}$; contradiction. We only need to find an $a \in E$ such that $\varphi'_a(a) \neq -\psi'(a)$. One can easily check that

$$\varphi_a = h_{-a} \circ (-\text{id}) \circ h_{h_a(\psi(a))} \circ h_a.$$

Direct calculations show that $\varphi'_a(a) = -\psi'(a) \iff a \in \mathbb{R}$. Thus it suffices to take any $a \in E \setminus \mathbb{R}$. \square

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REFERENCES

- [Cho] K. S. Choi, *Injective hyperbolicity of product domain*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. 5(1) (1998), 73–78.
- [Hah] K. T. Hahn, *Some remark on a new pseudo-differential metric*, Ann. Polon. Math. 39 (1981), 71–81.
- [Jar-Pfl] M. Jarnicki, P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter Exp. Math. 9, Walter de Gruyter, Berlin, 1993.
- [Ove] M. Overholt, *Injective hyperbolicity of domains*, Ann. Polon. Math. 62(1) (1995), 79–82.
- [Roy] H. L. Royden, *Remarks on the Kobayashi metric* in “Several complex variables, II”, Lecture Notes in Math. 189, Springer Verlag, 1971, pp. 125–137.
- [Two-Win] P. Tworzewski, T. Winiarski, *Continuity of intersection of analytic sets*, Ann. Polon. Math. 42 (1983), 387–393.

- [Ves] E. Vesentini, *Injective hyperbolicity*, Ricerche Mat. 36 (1987), 99–109.
[Vig] J.-P. Vigué, *Une remarque sur l'hyperbolicité injective*, Atti. Acc. Lincei Rend. fis. (8) 83 (1989), 57–61.

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